AN INFORMAL REPORT ON THE PRESENT STATE OF A THEORY OF OBJECTIVITY IN COMPARISONS

by

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I. Review of the Results

1. Applications of Models Suggested in 1960/1961.

In the publications [1] and [2] I suggested a class of stochastic models which were shown to be suitable for dealing with two main problems in test psychology, evaluation of individuals per se and population-independent comparisons of items. In practice these models have been tried out in several cases. Besides those reported in [ ] R. Brooks [4] applied them to results of testing school children with the Minnesota intelligence test (MMPI) and recently Erling B. Andersen [5] and [6] analyzed data on questionnaires from the field of social psychology. Apart from that a considerable amount of unpublished material awaits a suitable occasion for publication. Everywhere the models have proved to be very effective tools, also in cases where the data did not fit the model in question, often indicating, however, ways for further analysis and for further experiments.

2. Humanities and Natural Sciences.

When first suggesting the models I could offer no better excuse for them than their apparent suitability, which showed in their rather striking mathematical properties. In the paper [2] a more general point of view was indicated, according to which the models were strongly connected with what seemed to be basic demands for a much needed generalization of the concept of measurement. In continuation of that paper my attention was drawn to other fields of knowledge, such as economics, sociology, history, linguistics, evaluation of arts, etc. where claims are arising of being taken just as seriously as Natural Sciences.

At first sight the observational material in Humanities is very different from that in physics, chemistry and biology, not to speak of mathematics. But it might turn out that the difference is less essential than it would seem. In fact, the question is not whether the observations are of very different types, but whether Sciences could be firmly established on the basis of quite different types of observations.

3. Scientific Statements: Comparisons Being Objective.

Such considerations lead to the question from which I now start my inquiry: What is Science? Which conditions must be fulfilled when a statement can be qualified as scientific, thus competing with Natural Sciences?

That science should require observations to be measurable quantities is a mistake, of course; even in physics observations may be qualitative* - as in last analysis they always are**!

Two features seem indispensable in scientific statements: They deal with comparisons, and the comparisons must be objective. To complete these requirements I have to specify the kind of comparisons and the precise meaning of objectivity. When doing so I do not feel confident that all sorts of what can justifiably be called "science" are covered, but certainly a very large area of science is.

4. Specifying Comparisons.

Consider a class of "objects" to be mutually compared. The sense in which they should be compared is specified through a class of "agents", to each of which each object may be "exposed". On each exposure an "observation" - quantitative or qualitative is made. The whole set of such observations made when a finite number of objects $O_1, \ldots O_n$ are exposed to a finite number of agents $A_1, \ldots A_k$ form the data from which comparisons of the O's as regards their "reactions" to such agents as the A's can be inferred.

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* e.g. emission of radio-active particles observed as scintillations on a screen.

** e.g. reading off a point as located between two marks on a measuring rod.

5. Specifying Objectivity.

Now, within this framework, which I have taken from psychophysics, the "objectivity" of a comparative statement on, say, two objects, O_1 and O_2 is taken to mean that although being based upon the whole matrix of data it should be independent of which set of agents $A_1, \ldots, A_k$ out of the available class were actually used for the comparative purposes, and also of which objects $O_3, \ldots, O_n$, other than O_1 and O_2 were also exposed to the set of agents chosen.

6. Specific Objectivity, Some General Properties.

In order to distinguish this type of objectivity from other use of the same word I shall call it "specific objectivity", and in passing I beg you notice the relativity of this concept: it refers only to the framework specified by the class of objects, the class of agents and the kind of observations which define the comparison*.

Also the symmetric role of "objects" and "agents" should be observed, in consequence of which I confine my further analysis to simultaneous comparisons of objects and of agents on the basis of the same set of data.

Finally I wish to point out that in this context only the objects and/or the agents are subject to comparison, while the data themselves are not directly compared, they only serve as instruments for the comparisons aimed at.

The consequences of introducing these two concepts: (specific) comparisons and specific objectivity, completed by the requirement that a comparison is always possible and its result always unambiguous, are really overwhelming.

In presenting an outline of them I shall refrain from the greatest possible generality, mainly because it has not yet been

* This is a generalization of the invariance toward a specified group of transformations as required by Hermann Weyl [7] for objectivity in physics and even in mathematics.

fully explored, but also because certain stepwise specializations make things somewhat easier - and, even so, they may be difficult enough.

7. First Specialization: Parametrization and the Generality of It.

The first specialization is a parametrization of both objects, agents and observations.

As far as the comparison in question is concerned each object in the class considered is presumed to be fully characterized by a parameter θ which may be a real number or a vector consisting of a finite number, p, of elements, each being a real number. Similarly, each possible agent is presumed to be fully characterized by a parameter σ which is also a real vector of finite dimension, q. Finally, each contact between an object and an agent is in this context fully characterized by a parameter, ξ.

The last statement covers situations of widely differing types. In natural sciences events are often described in deterministic terms, as when a ball receiving a blow shows an acceleration which - but for errors of measurement - could be calculated from the mass of the ball and the force of the blow. However, in some cases, as in radio-active emissions, the outcome - viz. the number of so-called α-particles emitted in a given time - varies at random in repeated experiments, but this variation can be described in terms of a probability distribution (actually a Poisson distribution) that is governed by a certain parameter, the intensity of the emission multiplied by the time interval. Thus, if we want to compare a number of radio-active substances (the objects) by observing the number of emitted particles (the observations) in time intervals of different lengths (the agents) then the combination of each substance and each time interval can be fully characterized by a one-dimensional parameter.

In psychology and sociology observations are very often qualitative and the outcomes in repetitions variable. Attempts may then be made to describe the variation in terms of probability distributions, the mathematical form of which is the same in all the combinations of objects and agents in question, but each case then being fully characterized by a parameter, ξ.

In economics observations are often described as evaluations, expressed in terms of certain preferences which are connected through some sort of a "utility function".

In more recent statistical theory a concept of personal or subjective probability has made its appearence; it follows most of the ordinary probability axioms, but the probabilities are not confined to the interval $(0,1)$.

We shall consider deterministic and non-deterministic phenomena, represented in any way, having this feature in common that to each combination of object and agent is somehow attached a parameter ξ that is a real vector of a certain dimension, r.

8. Definition of Comparisons: Principle of Equivalent Agents (Objects).

The requirement that θ and σ fully characterize object and agent with respect to the comparison in question implies that ξ must be uniquely determined by θ and σ, i.e.,

(1) $\xi = \mu(\theta,\sigma)$ *

where μ is a univalued vector function of θ and σ.

The further requirement that the comparisons of θ's and σ's shall be unambiguous lead to demanding that the equation (1) for any given σ can be solved uniquely with respect to θ, and the other way round, i.e.,

(2) $\theta = \lambda(\xi,\sigma)$ and

(3) $\sigma = \kappa(\xi,\theta)$

are univalued functions of ξ and σ, resp. ξ and θ. In consequence ξ, θ and σ must be vectors of the same dimension**, i.e.

(4) $p = q = r.$

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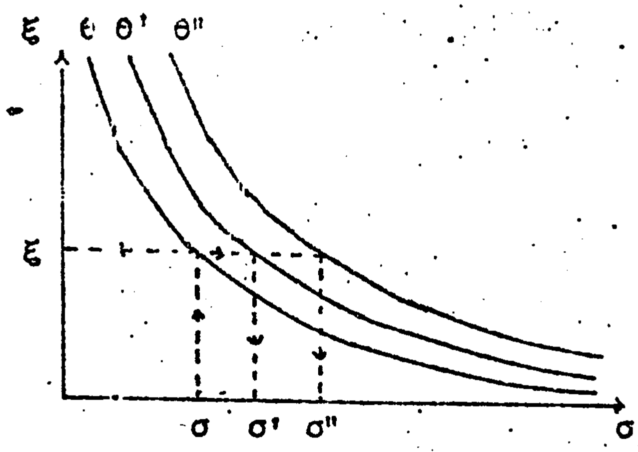
* (1) stands for r one-dimensional equations.

** If $p>r$ only r of the elements of θ could be determined from (1), and if $p<r$ the system would be overdetermined which would require a relation between the elements of ξ to hold, so that the dimension of ξ could be reduced.

At this stage the comparison concept aimed at can be clarly defined.

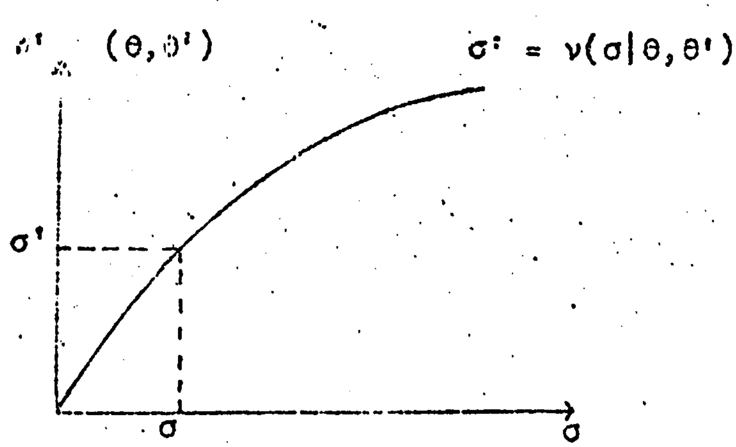
For illustration we take the case $r = 1$, i.e. all parameters are real numbers. In that case - adding that they may be restricted to the possitive axis - the equation (1) for three fixed objects may be represented by three monotonic curves:



To any chosen agent with parameters σ corresponds a unique observation parameter ξ, for the object with parameter θ. For another object with parameter θ' we may locate the agent with parameter σ' which produces the same ξ. Then the statement that σ' corresponds to σ yields a comparison (on the level ξ) of θ and θ', and when doing so for any σ we obtain a whole correspondence curve

(5) $$\sigma' = \nu(\sigma | \theta, \theta')$$

which yields a complete description of a comparison of θ and θ' based upon the principle of equivalent agents.



9. Transitivity and Translatability of Comparisons as Implied by Specific Objectivity.

Comparisons according to this principle are <u>transitive</u>: θ and θ'' may be compared directly or by taking comparisons of θ and θ'' with θ' as an intermediate stop. As seen from the first figure the result σ'' is the same. Thus one part of the specific objectivity requirement is fulfilled: <u>the comparison of any two objects is independent of which other objects enter into the comparison project.</u> In extreme consequence of the other part of the requirement, viz. the independence of which agents are chosen for establishing the comparison, the comparison can actually be carried out by means of one agent only. The result being unique it follows that any other choice must lead to the same comparative statement; however, as this statement amounts to presenting the whole curve, we must conclude that <u>the whole function (5) is fully determined by only one of its points (translatability)</u>.

What has been illustrated here in the case $r = 1$ holds any dimension of the parameters.

10. One-Dimensionality of Parameters Leading to Additivity (Specific Measurement).

In the special case of one-dimensional parameters a rather farreaching conclusion of the statement about the function (5) can be drawn, viz. that it is possible to choose the metrics in which ξ, θ and σ are expressed in such a way that (1) reduces to

$$\xi = \theta + \sigma, \tag{6}$$

i.e. the <u>reaction parameter is obtained simply by addition of the object parameter and the agent parameter</u>. In such cases we have thus obtained a measurement of as simple type as a psychometrician could wish. But it should be stressed that it has not been obtained through performing "the art of assigning numbers to observations" and see what happens.

It is based upon the validity of the rule (6), and whether that rule holds for a given kind of comparisons is a purely empirical question.

Thus, the recognition of a general law precedes the definition of a "specific" measurement, and this is just a situation that is characteristic for proper measuring (excluding ordering) in physics.

If the parameter dimension exceeds 1 the relation (6) no longer holds in full generality; it has to be replaced by a certain group theoretical relation, for the practical management of which adequate tools are still lacking.

11. Second Specialization: Observations Follwing Finite Probability. Distributions of the Same Form, but with Different Parameters.

We now proceed to the next step in the specialization process at which we shall assume that the observations are available, taking on "values" - quantitative or qualitative - which form a finite set

$$X : \{x^{(1)}, \ldots , x^{(m)}\} , \tag{7}$$

and that the variation as indicated above (p.4-5) can be described in terms of probability distributions of a common mathematical form, i.e.,

$$p \{x^{(h)} | \theta, \sigma\} = f_h(\xi), \quad h = 1, \ldots, m, \tag{8}$$

where ξ is a vector function (1) of θ and σ and where the real functions $f_h(\xi)$ are the same functions for all possible combinations of θ and σ. These functions, being probabilities, are non-negative and add up to unity for each ξ,

$$\sum_{h=1}^{m} f_h(\xi) = 1 . \tag{9}$$

12. Consequence for Dimension of Parameters.

Usually only a few observations are available for each (θ, σ)-combination, but in some cases - as in some types of psychophysical and biological reaction curve experiments - each (θ, σ) is represented by a large number of repetitions, thus allowing for a decent estimation of each of the probabilities (8).

As an extremely favourable case we may imagine that our data were these probabilities themselves, which then would be the only source available for information about ξ for any chosen (θ,σ)-combination. Accordingly it is a minimum requirement for carrying out the comparisons from the data that ξ for each (θ,σ) can be determined from the probabilities $p\{x^{(h)}|\theta,\sigma\}$, i.e. that the system (8) can be solved with respect to ξ.

Due to (9) there are really only m-1 equations, from which, therefore, no more than m-1 unknowns can be determined. Accordingly the vector ξ contains at most m-1 elements, i.e.,

(10) $$r \leq m-1 ,$$

an inequality that together with (4) sometimes restricts severely the possible dimensionalities of the object and agent parameters. If, in particular, only two response categories are available we must have

(11) $$r = 1 ,$$

i.e. the parameters must be one-dimensional, and in consequence the (ξ,θ,σ)-system, expressed in appropriate metrics, is additive.

13. Partial Utilization of Observational Categories.

Consider now in some detail how experimental or observational situations may turn out to be. Testing adaptability to certain exterior condition by means of a questionnaire of, say, 20 questions with 4 response categories, such as "good, fair, not too good, bad", may serve as a typical example. A well-chosen set of questions should range from strongly provocative in "positive direction" to strongly provocative in "negative direction". And similarly, a group of testees, well-chosen with a view to trying out the model decided upon, should comprise individuals of both excellent, poor and medium adaptation. If the social psychologist has succeeded in this design all four response categories will be used to a considerable extent. But of course, large parts of the observational matrix may be isolated where some of the categories, e.g., the positive ones, are not at all used. This may happen if the psychologist wishes to make a particular study of the answers to a group of negatively provocative questions by a group of persons - say, some school class of children - known or suspected to be rather unruly. We cannot expect such a truncated set of data to throw as much light upon the adequacy of the model or upon the parameters of individuals and items as the total set of data with its full variability. But it should be possible to conclude something, to carry out partial comparisons of individuals as well as of items. And we may require such comparisons, as far as they go, to be specifically objective!

14. Third Specialization: Complete Specific Objectivity, Definition

These considerations give rise to establishing a more restricted concept, which may be called complete specific objectivity. In order to explain this concept I consider a selection X' of categories out of the whole set (7), for instance $x^{(1)}$ and $x^{(2)}$, and in any actually observed matrix of data I keep those for which the observed category belongs to X' and leave all other data out of consideration. Through this mutilation some of the objects as well as some of the agents may be wholly trown out, but among those left over comparisons may be carried out, and we may require that such comparisons of objects and of agents as can be carried out on the basis of the mutilated data matrix shall be specifically objective. The complete specific objectivity is present when this holds for every possible selection X' out of the whole set of categories (7).

As indicated above we cannot expect as much information about the parameters from a mutilated matrix as from an undamaged one. This can be expressed in the following terms: The parameter ξ, θ and σ are related to the set X of categories, and if that is changed (reduced) to X' the parameters will also change, say to ξ', θ' and σ'. The complete specific objectivity requires that both θ' and σ' can be compared with specific objectivity for any choice of X'.

15. Vectorial Additivity of Parameters as Implied by Complete Specific Objectivity.

In particular, we may choose X' as any pair of categories $(x^{(\lambda)}, x^{(\mu)})$. In these cases $m' = 2$, therefore $r' = 1$, and, according to (6),

$$\xi' = \theta' + \sigma'. \tag{12}$$

It takes a bit of algebra to show that if (12) holds for any pair then the parameters ξ, θ and σ corresponding to the total set of categories can be expressed in such r-dimensional metrics that (6) holds as a vector addition. Spelling the vectors out in elements

$$\begin{aligned} \xi &= (\xi_1, \ldots, \xi_r) \\ \theta &= (\theta_1, \ldots, \theta_r) \\ \sigma &= (\sigma_1, \ldots, \sigma_r) \end{aligned} \tag{13}$$

this means that

$$\begin{aligned} \xi_1 &= \theta_1 + \sigma_1 \\ &\text{-------------} \\ \xi_r &= \theta_r + \sigma_r \end{aligned} \tag{14}$$

Thus, while the additivity principle only holds for $r = 1$ in case of specific objectivity, it holds generally in case of complete specific additivity, in which case, then, a multidimensional measurement has been established.

16. Generality and Limitations of Complete Specific Objectivity. An Outstanding Problem.

It may be mentioned that the complete specific objectivity as opposed to ordinary specific objectivity is by no means a trivality. In the theory of relativity and in quantum mechanics the completeness does not hold.

It should be noticed that the derivation of the above resul requires that a probability distribution is attached to each object-agent combination, but stochastic independence is not assumed. Accordingly, it covers certain cases of stochastic processes, but this field has not yet been explored.

It has been assumed that the set X of categories is finite, but it presents no difficulties to extend the concept of complete specific objectivity to the case of an infinite, but enumerable set of categories and obtain the corresponding result. The concept can also be defined in cases of non-numerable sets, e.g. all real numbers, but it has not yet been cleared up if also the general additivity principle holds.

17. Fourth Specialization: Stochastical Independence of All Observations (for Fixed Set of Parameters).

Turning now to the final specialization, I require that all observations in a data matrix are stochastically independent.

When taking this step we encounter a somewhat subtle problem as regards the exact meaning of the invariance demanded by the specific objectivity. The preceding sections dealing only with the parameters left us in no doubt: if one set of objects and agents yielded a set of ξ's from which it was concluded that $\theta_1 = 2\,\theta_2$, say, then exactly the same relation must obtain from any other possible set of agents and objects (including O_1 and O_2) if the statement be specifically objective (cf. the discussion of mass and force in [1], chapt. VII).

18. The Invariance Demanded by Specific Objectivity, Redefined as Statistical Equivalence.

However, when introducing now the stochastic independence we are also concerned with the observed data themselves, our point of departure being the joint probability of the whole set of observations which, due to stochastic independence, is simply the product of the probabilities of the single observations. From this probability a specifically objective conclusion should be drawn about θ_1 and θ_2, say. Primarily, this conclusion itself is a probability statement* involving no other parameters than θ_1 and θ_2, but utilizing the whole set of observations. Accordingly, even on repetition of the same confrontations of objects and agents such a statement may - and usually will - change. And still greater changes may be expected when - as the specific objectivity requires - the agents and the objects not to be compared are replaced by other ones. Therefore the invariance of

* Inferences in terms of hypothesis testing, confidence limits, etc., are secondary, derived from the probability statements.

the statement derived from various sets of data, as demanded by the specific objectivity, must not be construed as an identity in terms. But such statement may be statistically equivalent in the sense that the derived probabilities are compatible, whichever values θ_1 and θ_2 may have. Otherwise expressed, the estimates of the relation between θ_1 and θ_2 from various sets of data should not differ significantly from each other. As "significance" basically is a conventional concept this reference does not give an unambiguous answer to our invariance problem. The main point in this connection is, however, that it should be possible to test the hypothesis that the relation between θ_1 and θ_2 is the same in all cases considered, and that this test holds whichever values θ_1 and θ_2 may have; accordingly the test must be based upon a probability statement that is independent of θ_1 and θ_2, and therefore of all parameters. Thus, that such a probability statement can be found is anyhow a necessary condition for ascertaining the specific objectivity.

The final step to utilize such probabilities of actually testing the invariance hypothesis I may at present leave to the discretion of statisticians belonging to various schools - hoping, however, some day to return to the matter on a more objective basis.

19. Necessary and Sufficient Condition for Specific Objectivity in Case of Two Categories and Stochastic Independence.

In order now to realize the consequences of the additional requirements we shall first deal with the case of two response categories. The two probabilities corresponding to a (θ,σ)-combination may be written

$$p\{x^{(1)}|\theta,\sigma\} = f_1(\theta + \sigma)$$

$$p\{x^{(2)}|\theta,\sigma\} = f_2(\theta + \sigma) = 1 - f_1(\theta + \sigma)$$

(cf. (8), (9), and (6)), or, more conveniently

$$p\{x^{(1)}|\theta,\sigma\} = \frac{\mu(\theta+\sigma)}{1+\mu(\theta+\sigma)},$$

(15)

$$p\{x^{(2)}|\theta,\sigma\} = \frac{1}{1+\mu(\theta+\sigma)}.$$

Considering two values of θ and requiring that it should be possible to derive a (non-trivial) probability that is independent of σ and therefore may be used for comparing the θ's, there are in fact only a few possibilities, most of which can be ruled out by and by. Only one possibility is left over, viz.

(16) $$p\{x^{(1)}|\theta,\sigma\} = \frac{e^{\theta+\sigma}}{1+e^{\theta+\sigma}}$$

which in a slightly modified form

(17) $$p\{x^{(1)}|\xi,\varepsilon\} = \frac{\xi\varepsilon}{1+\xi\varepsilon}$$

was investigated in reference [1] and discussed at length in a recent paper [3].

The main result can be summarized as follows:

The validity of the model (17) is both necessary and sufficient for attaching specifically objective estimations of the parameters as well as specifically objective appraisals of the model as representing the data.

Technically, we have for each object O_ν to count the total number r_ν of $x^{(1)}$-responses to the agents $A_1,\ldots,A_k$ and for each agent A_i to count the total number s_i of $x^{(1)}$-responses from the objects $O_1,\ldots,O_n$. The conditional probability of $r_1,\ldots,r_n$ for given values of $s_1,\ldots,s_k$ depends only on the object parameter, while the conditional probability of $s_1,\ldots,s_k$ for given values of $r_1,\ldots,r_n$ depends only on the agent parameters. Finally, the conditional probability of the observed set of data, given the two sets of marginals, is independent of all the parameters and may, therefore, serve as a basis for appraising the model.

20. Necessary and Sufficient Condition for Complete Specific Objectivity in Case of Stochastic Independence and Finite Set of Categories.

For $m > 2$ I take recourse to the complete specific objectivity* obtaining for each pair of categories the result just mentioned for $m = 2$. On combining all these results I obtain the general result which briefly may be stated as follows:

In case of a finite or enumerable set of response categories the validity of the model presented as formula (3.2) with $\chi(x) = 0$ in reference [2] is the necessary and sufficient condition for obtaining complete specifically objective estimations of the parameters of objects and of agents as well as complete specific objectivity in appraising the model as representing the set of data.

21. Special Case: Maximal Dimension of Parameters.

In order to expound in some details the content of this statement I shall first consider the case where the dimension of the parameters is maximal, i.e.

(18) $$r = m - 1 .$$

In that case the model may be looked upon as a direct generalization of (17):

(19) $$p\{x^{(h)}|\xi,\varepsilon\} = \frac{\xi^{(h)}\varepsilon^{(h)}}{\gamma(\xi,\varepsilon)}$$

where $\xi^{(h)}$ and $\varepsilon^{(h)}$ are positive scalar parameters pertaining to the response category $x^{(h)}$ and where

(20) $$\gamma(\xi,\varepsilon) = \sum_{h=1}^{m} \xi^{(h)}\varepsilon^{(h)}$$

normalizes the right hand terms to make them add up to unity. The object and agent parameters ξ and ε are proportional to the vectors

(21) $$(\xi^{(1)},\ldots\ldots,\xi^{(m)}) \text{ and } (\varepsilon^{(1)},\ldots\ldots,\varepsilon^{(m)}).$$

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* A theory for ordinary specific objectivity has not yet been worked out.

If desired $\xi^{(m)}$ and $\varepsilon^{(m)}$ may be put equal to 1, and (19) then reduces directly to (17) for m = 2.

22. Minimal Dimension of Parameters.

If

(22) $r < m-1$

the representation (19) still holds, but then of course $\xi^{(1)},\ldots\ldots,\xi^{(m-1)}$ as well as $\varepsilon^{(1)},\ldots\ldots,\varepsilon^{(m-1)}$ must be functionally related. When the complete specificity is taken fully into account it may be shown that these relationships are logarithmically linear. They are simplest in case r = 1. Then the parameters proper of objects and agents are one-dimensional and we have

(23) $$\begin{cases} \log \xi^{(h)} = \alpha^{(h)} + \varphi^{(h)}\theta \\ \log \varepsilon^{(h)} = \beta^{(h)} + \varphi^{(h)}\sigma \end{cases},$$

where

(24) $$\alpha^{(h)} + \beta^{(h)} = \tau^{(h)}$$

and $\varphi^{(h)}$ are coefficients that are characteristic for the categories, irrespective of the parameters. The model (19) then takes the form

(25) $$p\{x^{(h)}|\theta,\sigma\} = \frac{1}{\gamma(\theta,\sigma)} \exp \left(\varphi^{(h)}(\theta+\sigma) + \tau^{(h)}\right)$$

where now

(26) $$\gamma(\theta,\sigma) = \sum_{h=1}^{m} \exp. \left(\varphi^{(h)}(\theta+\sigma) + \tau^{(h)}\right)$$

This is the simplest case of what has recently been christened Models for Measuring.

23. Submaximal Dimension of Parameters.

If r exceeds 1, but is less than m-1 the formula (23) may be such interpreted that it still holds. $\varphi^{(h)}$ as well as the parameters θ and σ then are r-dimensional vectors:

$$(27) \quad \begin{cases} \varphi^{(h)} = (\varphi_1^{(h)},\ldots\ldots,\varphi_r^{(h)}), \\ \theta \quad = (\theta_1 \quad ,\ldots\ldots,\theta_r) \\ \sigma \quad = (\sigma_1 \quad ,\ldots\ldots,\sigma_r) \end{cases}$$

and the products $\varphi^{(h)}\theta$ and $\varphi^{(h)}\sigma$ should be read as "the inner product of the vectors":

$$(28) \quad \begin{cases} \varphi^{(h)}\theta = \varphi_1^{(h)}\theta_1 + \ldots\ldots + \varphi_r^{(h)}\theta_r \\ \varphi^{(h)}\sigma = \varphi_1^{(h)}\sigma_1 + \ldots\ldots + \varphi_r^{(h)}\sigma_r \end{cases} .$$

The sum $\varphi + \sigma$ is taken to be the vectorial sum (14).

24. Relation and Non-Relation to Factor Analysis.

The structure (23), (28) reminds of the structural part of the factor analysis specification and it is subject to the same sort of inherent indeterminacy. But otherwise the model for measuring differs fundamentally from factor analysis.

As a main point the linear structure (23), (28) is not assumed for the observations, but for the essential part of the logarithms of the probabilities. For technical reasons it may be added that in contrast to the factor analysis specification our model does not imply supplementary "error terms".

25. Selection Vectors and Response Vectors.

As regards technicalities two points deserve special mention: First an analogy to the estimation of ξ's and ε's in the case $m = 2$ where we had to count the number of $x^{(1)}$-responses in two directions. With $m > 2$ we may indicate the response $x^{(h)}$ by a "selection vector"

$$(29) \qquad (0,\ldots\ldots,1,\ldots\ldots,0)$$

where all elements except the h:th are 0.

Then for each (O_ν, A_i)-combination such a selection vector

$$(30) \qquad a_{\nu i} = (a_{\nu i}^{(1)},\ldots\ldots,a_{\nu i}^{(h)},\ldots\ldots,a_{\nu i}^{(m)})$$

is observed and the observed set of data may be arranged as a matrix with selection vectors as elements.

Now for each agent A_i the n selection vectors may be added up - element by element - to a total response vector s_i and similarly for each object O_ν the k selection vectors are added up to a total vector r_ν. A basic theorem tells that, whatever r be, the probability of the response vectors $r_1,\ldots\ldots,r_n$ for given values of $s_1,\ldots\ldots,s_k$ depends on the vectors $\xi_1,\ldots\ldots,\xi_n$ and not upon $\varepsilon_1,\ldots\ldots,\varepsilon_k$. And, similarly, the probability of $s_1,\ldots\ldots,s_k$ for given values of $r_1,\ldots\ldots,r_n$ depends on the vectors $\varepsilon_1,\ldots\ldots,\varepsilon_k$, but not on $\xi_1,\ldots\ldots,\xi_n$. Finally the probability of the whole observed matrix of selection vectors, given both marginals $r_1,\ldots\ldots,r_n$ and $s_1,\ldots\ldots,s_k$ is independent of all of the parameters. Thus the instruments are available for deriving specifically objective statements about the ξ's, the ε's and the model, as representing the data.

ν \ i	Agents: .... i ....k	Total
Objects: 1 ⋮ ν ⋮ n	.... $(a_{\nu i}^{(1)},\ldots,a_{\nu i}^{(m)})$...	$(r_\nu^{(1)},\ldots,r_\nu^{(m)})=r_\nu$
Total	.... $(s_i^{(1)},\ldots,s_i^{(m)})$ $=s_i$...	

26. Estimation of Parameters: The Scoring Function and Structural Characteristic.

The next point is concerned with the derivation of estimates of the proper parameters θ and σ from the r_ν's and the s_i's.

The r_ν's, given the s_i's, serve to estimate $\xi_1,\ldots\ldots,\xi_n$ and according to (23) it then should be possible also to estimate the deviations of $\theta_1,\ldots\ldots,\theta_n$ from their average provided the $\varphi^{(h)}$'s were known. And on the same proviso the s_i's, given the r_ν's should yield estimates of the deviations of $\sigma_1, \ldots\ldots, \sigma_k$ from their average.

This conclusion attaches a particular significance to the $\varphi^{(h)}$'s.

In case the parameters are one-dimensional, $\varphi^{(h)}$ is a numerical value corresponding to the category $x^{(h)}$, so the set of them may be said to form a <u>quantification of the set of categories</u>. This quantification, however, cannot be chosen at will, it belongs to the structure of the model, but when available and combined adequately with the r_ν's and the s_i's we shall obtain the best possible (sufficient) estimates of the parameters of the model.

Therefore the $\varphi^{(h)}$-values may be considered as scorings of the categories but to distinguish them from more or less arbitrarily chosen scores we may term the set of them the <u>specific scoring function</u>.

When the dimension of the parameters exceeds 1 the situation is in principle much the same, only that the specific scoring function is now a vector function - or otherwise put, the elements $\varphi_j^{(h)}$, $h = 1, \ldots\ldots, m$, $j = 1, \ldots\ldots, r$ form <u>the specific scoring matrix</u>.

27. Scoring Function as Part of the Model.

In some - presumably rare - cases, of which [1] chapter II,1 is an example, the scoring function - and also the more secondary "structural characteristic" $\tau^{(h)}$ - can be derived from theoretical considerations. But in general a solid basis for even a fair guess is hardly present, I am afraid, and the question then arises: how to get hold of them. With a view to the strict and general principles laid down here this presents a serious problem.

The whole argumentation aims at arriving at objective evaluations, but only of the objects and the agents, i.e. the θ's and the σ's, but nothing else.

In this context the φ's and the τ's belong to the model, on the basis of which the objectivity is attained. The model itself is <u>not estimated</u>, it is only <u>suggested</u>, for what it is worth, as a necessary condition for the desired objectivity, but it can be tested if it is worth anything. And this also holds as regards any suggested scoring function.

28. Estimation of Scoring Function and Structural Characteristic.

As indicated in ref. [2] we may, however, go somewhat further. We may, in fact, estimate the scoring function directly from the data and in the papers [5] and [6] by Erling D. Andersen one possible procedure has been worked out in details and programmed in Algol.

29. Non-Objectivity of Estimates.

This estimation, however, is based upon an attempt at separating the θ's and τ's from the θ's and the σ's, but it is not possible to achieve that completely. We can obtain some sort of "unbiased estimates", but "the precision" of them will depend on the parameters θ and σ. Therefore the estimation of the φ's and the τ's cannot be specifically objective.

30. Relaxation to Almost Specific Objectivity?

In practice, on the other hand, the estimations seem to work perfectly satisfactory. Why that is so, has not yet been cleared up. At present my conjecture is that although the standard errors, say, of the estimates do depend on the θ's and σ's, a fairly close upper bound for them can be found which is independent of the unwarranted parameters. This conjecture points to a possible relaxation of our basic concept to some sort of "almost specific objectivity" - the development of which, however, wholly belongs to the future.

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